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COMPILATION OF RADIATION SHAPE FACTORS
FOR CYLINDRICAL ASSEMBLIES

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ABSTRACT

The design and evaluation of cylindrical furnaces, kilns, reactors, and other devices in which radiant heat transfer is important requires knowledge of shape factors in order to assess the energy transfer by radiation. Analytical expressions for the shape factors of various cylindrical assemblies, some heretofore unavailable, have been obtained and tabulated.

COMPILATION OF RADIATION SHAPE FACTORS FOR CYLINDRICAL ASSEMBLIES

By H. Leuenberger and R. A. Person

Introduction

For the design and evaluation of furnaces, kilns, reactors, driers, and other process equipment of cylindrical shape, in which radiant heat interchange can be significant, it is advantageous to have accurate and complete knowledge of geometric shape factors to assess reliably the radiant heat interchange.

The technical literature contains values for a number of shape factors, some in the form of analytical expressions and some as numerical approximations or graphical solutions. A large proportion of this information was obtained for assemblies of semi-infinite or differential dimensions, which can be dealt with fairly easily. The analytical treatment of the technically important configurations of finite dimensions is more complicated, and recourse has usually been had to approximations only.

Analytical and graphical presentations of shape factors for certain elementary configurations were initially given by Hottel (1,2,3)¹. These factors or their equivalents have since appeared in many heat transfer texts (4,5) and elsewhere, together with numerous examples of their use. The recognized need for reliable shape factor data for additional assemblies led the National Advisory Committee for Aeronautics to sponsor further study (6). While the available coverage of elementary configurations has been wide, few cylindrical assemblies have been treated, and the results are not conveniently available. Again most of this information is given in the form of numerical tabulations and graphs.

The present work on shape factors from the beginning aimed at the obtaining of, if possible, exact analytical expressions for shape factors of cylindrical

¹ Numbers in parentheses refer to the bibliography.

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assemblies of finite dimensions. This approach was motivated by the desire to minimize the tedium attendant to the compiling of numerical tabulations and to subsequent interpolations.

The field of shape factors covered in the present work somewhat overlaps that treated by others. Their work is thus generalized as well as supplemented by the compact analytical expressions of the present work, which have been derived and summarized for cylindrical assemblies of practical interest. The present work also fulfills the objective of presenting these factors in one compilation.

Evaluation of Shape Factors

The term "shape factor", or its synonyms, "view factor", "configuration factor", "form factor", "F factor", as related to radiant energy transfer between two surfaces, is defined as the fraction of the total radiation leaving the one surface in all directions which is intercepted by the other surface. The shape factor is a function of the shapes and relative position of the two surfaces under consideration. The above definition also covers the special case of the shape factor of a surface itself.

The method of deriving a shape factor for a pair of surfaces, as explained in detail in textbooks (4,7) starts by considering the radiation between differential elements of each surface. From the differential form the shape factor of finite assembly is, in general, arrived at by a quadruple integration. In special cases considerations of symmetry can limit the process to three integrations.

In setting up the differential form of a shape factor, the following laws of geometrical optics are assumed to apply:

1. The cosine law.
2. The inverse square law of radiation intensity.

Most engineering materials conform quite closely to these relations. Deviations are discussed in the literature (5,8).

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An important and useful consequence obtained from the definition of the shape factor is that the algebraic sum of all the shape factors from one surface to other surfaces in the assembly is unity. Another useful relation is that the product of the view factor from one surface to a second surface and the area of the one surface must equal the product of the view factor from the second to the first surface and the area of the second. These relations imply that, after a single shape factor between two surfaces of a configuration has been obtained by integration, shape factors for other surfaces of the same or kindred configurations can often be obtained by simple algebraic manipulations. This latter procedure has been termed "flux algebra" (6).

In addition to the above methods of evaluating shape factors, there are available various techniques of descriptive geometry (9), optical projection (8), and mechanical integration (2,6). However, the reported techniques evaluate the shape factor for cases where one area is of differential dimensions.

Tabulation of Shape Factors

The tabulation on the following pages can be considered a dictionary of shape factors for cylindrical assemblies. Each case deals with one configuration exhaustively, i.e., all shape factors, including limiting values, pertaining to it are given explicitly or can be arrived at by "flux algebra".

The configurations tabulated are:

- I Directly opposed parallel circular discs of unequal radii.
- II Cylinder of finite length.
- III Directly opposed parallel annuli of unequal radii.
- IV Cylinder and annulus contained in top.
- V Two concentric cylinders of equal radii, one above the other.
- VI Two concentric cylinders of equal length, one contained within the other.
- VII Cylinder and plane of equal length parallel to cylinder axis, plane outside cylinder.

VIII Cylinder and plane of equal length parallel to cylinder axis, plane inside cylinder.

IX Two concentric cylinders of unequal radii, one atop the other.

X Two concentric cylinders of unequal length, one enclosed by the other.

The integrated results for case I have been previously given in the literature (2). The finite area view factors for cases II, III, IV, V and part of IX are directly obtainable from I by "flux algebra". Cases II and V have been specifically discussed in the literature (10,11). Cases VI, VIII and part of IX required new integrations, analytical solutions for which apparently have not been obtained heretofore. Selected values for case VI had been calculated previously by approximate numerical methods (6). Case X follows directly from VI by "flux algebra". The results reported in case VII are obtainable by application of the cosine law to case VI.

In the tabulation the notation $F_{a \rightarrow b}$ is used to denote the shape factor from surface "a" to surface "b". Surfaces and geometric parameters are identified separately in each case. Values of inverse trigonometric functions are restricted to the interval 0 to π .

Expressions in closed form of the shape factors were obtained for all the configurations except cases VII, VIII and certain geometric ratios of IX. For illustration, selected numerical values for the shape factors of case VI are included as graphs.

The expressions of the above exceptions are included for completeness. They contain one integral which may be conveniently approximated numerically by a simple averaging procedure.

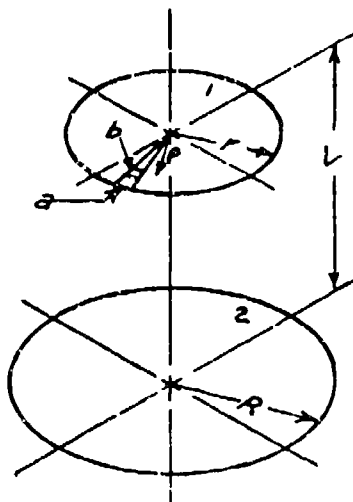
Expressions for the shape factor from a differential element of one surface are also given so that the shape factor for an arbitrary portion of one surface may be approximated by considering a sufficient number of differential elements within the restricted domain. The shape factors from a differential surface may also be used to numerically approximate factors for other assemblies than those

tabulated, e.g., shape factors for differential strips of cases VII and VIII may be used to obtain shape factors for finite surfaces of non-concentric cylinders of equal length, one within the other. The factors for differential elements were obtained by integration, "flux algebra", or by differentiation of shape factors for finite areas.

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I Directly Opposed Parallel Discs



- 1 upper disc
- 2 lower disc
- r radius of upper disc
- R radius of lower disc
- L distance between discs
- a differential sector of 1.
- b differential element of a
- ρ distance between b and disc axis

$$F_{b \rightarrow 2} = \frac{1}{2} \left[1 - \frac{\rho^2 - R^2 + L^2}{\sqrt{(\rho^2 + R^2 + L^2)^2 - 4\rho^2 R^2}} \right]$$

$$F_{a \rightarrow 2} = F_{1 \rightarrow 2} = \frac{1}{2} \left[1 + \frac{R^2 + L^2}{r^2} - \sqrt{\left(1 + \frac{R^2 + L^2}{r^2}\right)^2 - 4\frac{R^2}{r^2}} \right] *$$

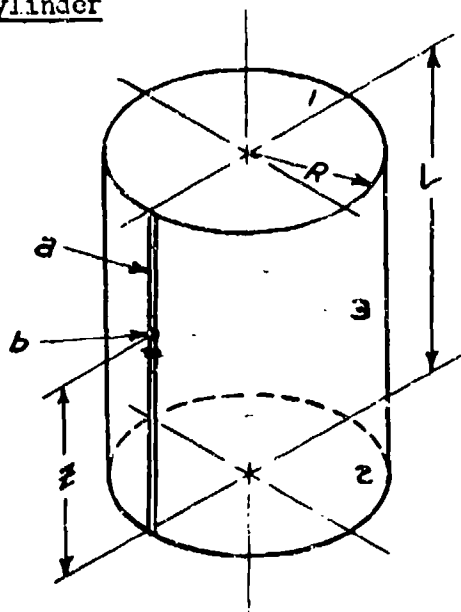
$$\text{when } R = r, F_{1 \rightarrow 2} = 1 - \frac{1}{2} \left(\frac{L}{R} \sqrt{4 + \frac{L^2}{R^2}} - \frac{L^2}{R^2} \right)$$

Limiting values

$L \rightarrow 0:$	$F_{1 \rightarrow 2} = 1$	$R \rightarrow r$
	$F_{1 \rightarrow 2} = 1^2/r^2$	$R \leftarrow r$
$L \rightarrow \infty:$	$F_{1 \rightarrow 2} = 0$	
$R \rightarrow 0:$	$F_{1 \rightarrow 2} = 0$	
$R \rightarrow \infty:$	$F_{1 \rightarrow 2} = 1$	
$r \rightarrow 0:$	$F_{1 \rightarrow 2} = R^2/(L^2 + R^2)$	
$r \rightarrow \infty:$	$F_{1 \rightarrow 2} = 0$	

* Equivalent to expression given by Hottel in McAdams, "Heat Transmission", 2nd ed., p. 54, McGraw-Hill, 1942.

II Cylinder



- 1 top of cylinder
- 2 bottom of cylinder
- 3 curved surface of cylinder
- R radius of cylinder
- L height of cylinder
- a differential vertical strip of 3
- b differential element of a
- z distance between b and 2

$$F_{b \rightarrow 2} = \frac{z^2 + 2R^2}{2R\sqrt{z^2 + 4R^2}} - \frac{z}{2R}$$

$$F_{b \rightarrow 3} = 1 + \frac{L}{2R} - \frac{z^2 + 2R^2}{2R\sqrt{z^2 + 4R^2}} - \frac{(L - z)^2 + 2R^2}{2R\sqrt{(L - z)^2 + 4R^2}}$$

$$F_{1 \rightarrow 3} = F_{2 \rightarrow 3} = \frac{1}{2} \left(\frac{L}{R} \sqrt{4 + \frac{L^2}{R^2}} - \frac{L^2}{R^2} \right)$$

$$F_{a \rightarrow 2} = F_{3 \rightarrow 2}$$

$$F_{a \rightarrow 3} = F_{3 \rightarrow 3} = 1 - \frac{1}{2} \left(\sqrt{4 + \frac{L^2}{R^2}} - \frac{L}{R} \right)$$

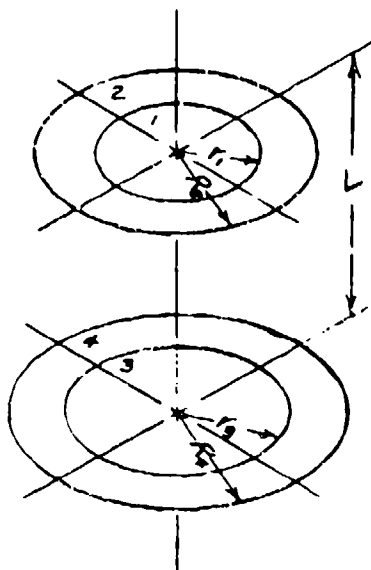
Limiting values:

$L \rightarrow \infty$	$F_{1 \rightarrow 3} = 1$
	$F_{3 \rightarrow 3} = 1$
$L \rightarrow 0$	$F_{1 \rightarrow 3} = 0$
	$F_{3 \rightarrow 3} = 0$
$R \rightarrow \infty$	$F_{1 \rightarrow 3} = 0$
	$F_{3 \rightarrow 3} = 0$
$R \rightarrow 0$	$F_{1 \rightarrow 3} = 1$
	$F_{3 \rightarrow 3} = 1$

$$z \rightarrow 0: F_{b \rightarrow 2} = \frac{1}{2}$$

$$F_{b \rightarrow 3} = \frac{1}{2} + \frac{L}{2R} - \frac{L^2 + 2R^2}{2R\sqrt{L^2 + 4R^2}}$$

III Directly Opposed Parallel Annuli



- 1 inner disc contained by top annulus
- 2 top annulus
- 3 inner disc contained by bottom annulus
- 4 bottom annulus
- r inner radius of annulus
- R outer radius of annulus
- L distance between annuli
- $R_2 > r_1$, $R_4 > r_3$

$$F_{1 \rightarrow 4} = \frac{1}{2} \left[\frac{R_4^2 - r_3^2}{r_1^2} - \sqrt{\left(1 + \frac{R_4^2 + L^2}{r_1^2}\right)^2 - 4 \frac{R_4^2}{r_1^2}} + \sqrt{\left(1 + \frac{R_4^2 + L^2}{r_1^2}\right)^2 - 4 \frac{r_3^2}{r_1^2}} \right]$$

$$F_{2 \rightarrow 4} = \frac{1}{2(R_2^2 - r_1^2)} \left[\sqrt{(R_2^2 + r_3^2 + L^2)^2 - (2r_3 R_2)^2} - \sqrt{(R_2^2 + R_4^2 + L^2)^2 - (2R_2 R_4)^2} \right. \\ \left. + \sqrt{(r_1^2 + R_4^2 + L^2)^2 - (2r_1 R_4)^2} - \sqrt{(r_1^2 + r_3^2 + L^2)^2 - (2r_1 r_3)^2} \right]$$

When $R_2 = R_4 = R$, and $r_1 = r_3 = r$

$$F_{1 \rightarrow 4} = \frac{1}{2} \left[\frac{R^2 - r^2}{r^2} - \sqrt{\left(1 + \frac{R^2 + L^2}{r^2}\right)^2 - 4 \left(\frac{R}{r}\right)^2} + \frac{L}{R} \sqrt{4 + \frac{L^2}{R^2}} \right]$$

$$F_{2 \rightarrow 4} = \left(\frac{R^2}{R^2 - r^2} \right) \left[\sqrt{\left(1 + \frac{r^2 + L^2}{R^2}\right)^2 - 4 \left(\frac{r}{R}\right)^2} - \frac{L}{2R} \left(\sqrt{4 + \frac{L^2}{R^2}} + \sqrt{4 \frac{r^2}{R^2} + \frac{L^2}{R^2}} \right) \right]$$

Limiting values:

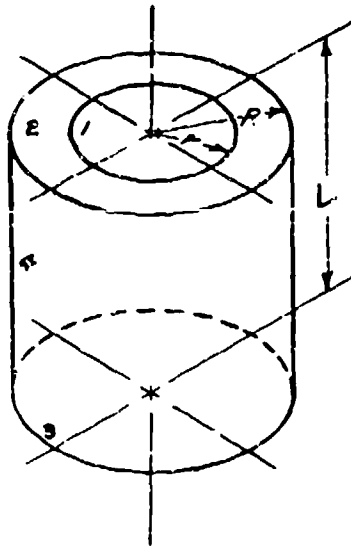
$L \rightarrow \infty$:	$F_{1 \rightarrow 4} = 0$	
	$F_{2 \rightarrow 4} = 0$	
$L \rightarrow 0$:	$F_{1 \rightarrow 4} = 0$	$r_3 > r_1$
	$= 1 - r_3^2/r_1^2$	$R_4 > r_1 > r_3$
	$= \frac{R_4^2 - r_3^2}{r_1^2}$	$r_1 > R_4$

$$\begin{aligned}
 L \rightarrow 0: \quad F_{2 \rightarrow 4} &= 0 \\
 &= \frac{R_2^2 - r_1^2}{R_2^2 - r_1^2} \\
 &= 1 \\
 &= \frac{R_2^2 - r_1^2}{R_2^2 - r_1^2}
 \end{aligned}
 \quad
 \begin{aligned}
 r_3 > R_2 \text{ or } r_1 > R_4 \\
 R_4 > R_2 > r_3 > r_1 \\
 R_4 > R_2 \text{ and } r_1 > r_3 \\
 R_2 > R_4 \text{ and } r_1 > r_3
 \end{aligned}$$

$$R_2 \rightarrow r_1: F_{2 \rightarrow 4} = \frac{1}{2} \left[\frac{r_1^2 - r_3^2 + L^2}{\sqrt{(r_1^2 + r_3^2 + L^2)^2 - (2r_3 r_1)^2}} - \frac{r_1^2 - R_4^2 + L^2}{\sqrt{(r_1^2 + R_4^2 + L^2)^2 - (2R_4 r_1)^2}} \right]$$

$$r_1 \rightarrow 0: F_{1 \rightarrow 4} = \frac{R_4^2}{L^2 + R_4^2} - \frac{r_3^2}{L^2 + r_3^2}$$

IV Cylinder and Annulus Contained in Top



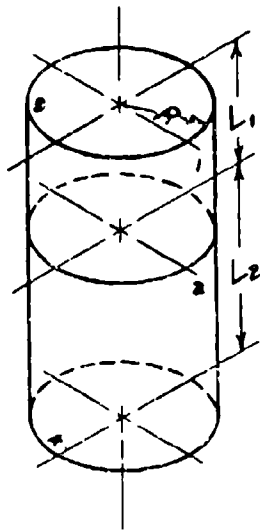
- 1 disc contained in cylinder top
- 2 annulus in cylinder top
- 3 bottom of cylinder
- 4 curved surface of cylinder
- r inner radius of annulus
- R radius of cylinder
- L height of cylinder

$$\begin{aligned}
 F_{2 \rightarrow 4} &= \frac{1}{2} \left[1 + \left(\frac{1}{R^2 - r^2} \right) \left(L \sqrt{4R^2 + L^2} - \sqrt{(r^2 + R^2 + L^2)^2 - (2rR)^2} \right) \right] \\
 F_{1 \rightarrow 4} &= \frac{1}{2} \left[1 - \frac{R^2 + L^2}{r^2} + \sqrt{\left(1 + \frac{R^2 + L^2}{r^2} \right)^2 - 4 \frac{R^2}{r^2}} \right]
 \end{aligned}$$

Limiting values:

$$r \rightarrow R: F_{2 \rightarrow 4} = \frac{1}{2} \left(1 + \frac{L}{\sqrt{4R^2 + L^2}} \right)$$

V Two Concentric Cylinders of Equal Radii, One Above the Other



- 1 curved surface of top cylinder
- 2 curved surface of bottom cylinder
- 3 top of cylinder 1
- 4 bottom of cylinder 2
- R radius of cylinders
- L_1 height of top cylinder
- L_2 height of bottom cylinder

$$F_{1 \rightarrow 4} = \frac{1}{4} \left[\left(\frac{L_1 + L_2}{L_1} \right) \sqrt{L_1^2 + \left(\frac{L_1 + L_2}{R} \right)^2} - \frac{L_1 + 2L_2}{R} - \frac{L_2}{L_1} \sqrt{L_1^2 + \frac{L_2^2}{R^2}} \right]$$

$$F_{1 \rightarrow 2} = \frac{L_2}{2R} + \frac{1}{4} \left[\sqrt{L_1^2 + \frac{L_2^2}{R^2}} + \frac{L_2}{L_1} \sqrt{L_1^2 + \frac{L_2^2}{R^2}} - \left(\frac{L_1 + L_2}{L_1} \right) \sqrt{L_1^2 + \left(\frac{L_1 + L_2}{R} \right)^2} \right]$$

when $L_1 = L_2 = L$

$$F_{1 \rightarrow 4} = \frac{1}{4} \left[4 \sqrt{1 + \frac{L^2}{R^2}} - \frac{3L}{R} - \sqrt{L^2 + \frac{L^2}{R^2}} \right]$$

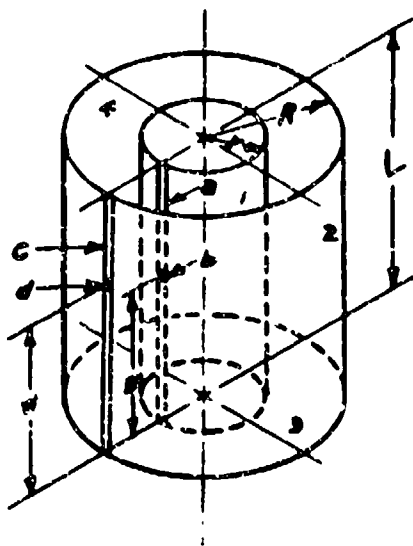
$$F_{1 \rightarrow 2} = \frac{1}{2} \left[\frac{L}{R} + \sqrt{L^2 + \frac{L^2}{R^2}} - 2 \sqrt{1 + \frac{L^2}{R^2}} \right]$$

Limiting values:

$$L_1 \rightarrow 0 : F_{1 \rightarrow 4} = \frac{L_2^2 + 2R^2}{2R\sqrt{L_2^2 + 4R^2}} - \frac{L_2}{2R}$$

$$F_{1 \rightarrow 2} = \frac{1}{2} + \frac{L_2}{2R} - \frac{L_2^2 + 2R^2}{2R\sqrt{L_2^2 + 4R^2}}$$

VI

Two Concentric Cylinders of Equal Length, One Contained Within the Other

- 1 curved exterior surface of inner cylinder
- 2 curved interior surface of outer cylinder
- 3 bottom annulus contained between 1 and 2
- 4 top annulus contained between 1 and 2
- R radius of outer cylinder
- r radius of inner cylinder
- L height of cylinders
- a differential vertical strip of 1
- b differential element of a
- c differential vertical strip of 2
- d differential element of c
- W distance between b and 3
- W distance between d and 3

$$F_{d-1} = \frac{F}{R} \left\{ 2 - \frac{1}{\pi} \left(\cos^{-1} \frac{W^2 - R^2 + r^2}{W^2 + R^2 - r^2} + \cos^{-1} \frac{(L-W)^2 - R^2 + r^2}{(L-W)^2 + R^2 - r^2} \right. \right. \\ \left. - \frac{W}{r} \left[\frac{W^2 + R^2 + r^2}{\sqrt{(W^2 + R^2 + r^2)^2 - 4r^2R^2}} \cos^{-1} \frac{r(W^2 - R^2 + r^2)}{R(W^2 + R^2 - r^2)} \right] \right. \\ \left. - \left(\frac{L-W}{r} \right) \left[\frac{(L-W)^2 + R^2 + r^2}{\sqrt{[(L-W)^2 + R^2 + r^2]^2 - 4r^2R^2}} \cos^{-1} \frac{r[(L-W)^2 - R^2 + r^2]}{R[(L-W)^2 + R^2 - r^2]} \right] \right. \\ \left. + \frac{L}{r} \cos^{-1} \frac{r}{R} \right\}$$

$$F_{d-2} = 2 \left(1 - \frac{r}{R} \right) + \frac{L}{2R} - \frac{W^2 + 2R^2}{2R\sqrt{4R^2 + W^2}} - \frac{(L-W)^2 + 2R^2}{2R\sqrt{4R^2 + (L-W)^2}} \\ + \frac{1}{\pi} \left(\frac{2r}{R} \left(\tan^{-1} \frac{2\sqrt{R^2 - r^2}}{W} + \tan^{-1} \frac{2\sqrt{R^2 - r^2}}{L-W} \right) + \frac{L}{R} \sin^{-1} \left(1 - \frac{2r^2}{R^2} \right) \right) \\ - \frac{W^2 + 2R^2}{R\sqrt{4R^2 + W^2}} \frac{\sin^{-1} \frac{4(R^2 - r^2)}{W^2 + 4(R^2 - r^2)} + \frac{W^2}{R^2} (R^2 - 2r^2)}{W^2 + 4(R^2 - r^2)} \\ - \frac{(L-W)^2 + 2R^2}{R\sqrt{4R^2 + (L-W)^2}} \frac{\sin^{-1} \frac{4(R^2 - r^2)}{(L-W)^2 + 4(R^2 - r^2)} + \frac{(L-W)^2}{R^2} (R^2 - 2r^2)}{(L-W)^2 + 4(R^2 - r^2)} \Bigg\}$$

$$F_{d \rightarrow 3} = \frac{1}{2} - (F_{d \rightarrow 1}^* + F_{d \rightarrow 2}^*)$$

$$F_{b \rightarrow 3} = \frac{1}{2\pi} \left\{ \cos^{-1} \frac{L^2 - R^2 + r^2}{L^2 + R^2 - r^2} - \frac{R}{L} \left[\frac{L^2 + R^2 + r^2}{(L^2 + R^2 + r^2)^2 - 4R^2L^2} \cos^{-1} \frac{r(L^2 - R^2 + r^2)}{R(L^2 + R^2 - r^2)} - \cos^{-1} \frac{r}{R} \right] \right\}$$

$$F_{b \rightarrow 2} = 1 - (F_{b \rightarrow 3} + F_{b \rightarrow 4})$$

$$F_{2 \rightarrow 1} = F_{c \rightarrow 1} = \frac{r}{R} \left\{ 1 - \frac{1}{\pi} \left[\cos^{-1} \frac{L^2 - R^2 + r^2}{L^2 + R^2 - r^2} - \frac{1}{2rL} \left[\sqrt{(L^2 + R^2 + r^2)^2 - (2rR)^2} \cos^{-1} \frac{r(L^2 - R^2 + r^2)}{R(L^2 + R^2 - r^2)} + (L^2 - R^2 + r^2) \sin^{-1} \frac{r}{R} - \frac{\pi}{2} (L^2 + R^2 - r^2) \right] \right] \right\} **$$

$$F_{2 \rightarrow 2} = F_{c \rightarrow 2} = 1 - \frac{r}{R} + \frac{1}{\pi} \left\{ \frac{2r}{R} \tan^{-1} \frac{2\sqrt{R^2 - r^2}}{L} - \frac{L}{2R} \left[\frac{\sqrt{4R^2 + L^2}}{L} \sin^{-1} \frac{4(R^2 - r^2) + \frac{L^2}{R^2}(R^2 - 2r^2)}{L^2 + 4(R^2 - r^2)} - \sin^{-1} \frac{R^2 - 2r^2}{R^2} + \frac{\pi}{2} \left(\frac{\sqrt{4R^2 + L^2}}{L} - 1 \right) \right] \right\} **$$

$$F_{2 \rightarrow 3} = F_{c \rightarrow 3} = \frac{1}{2} (1 - F_{2 \rightarrow 1} - F_{2 \rightarrow 2})$$

$$F_{3 \rightarrow 4} = 1 - \frac{1}{2} \left(\frac{L}{R^2 - r^2} \right) [r - R(F_{2 \rightarrow 2} + 2F_{2 \rightarrow 1} - 1)]$$

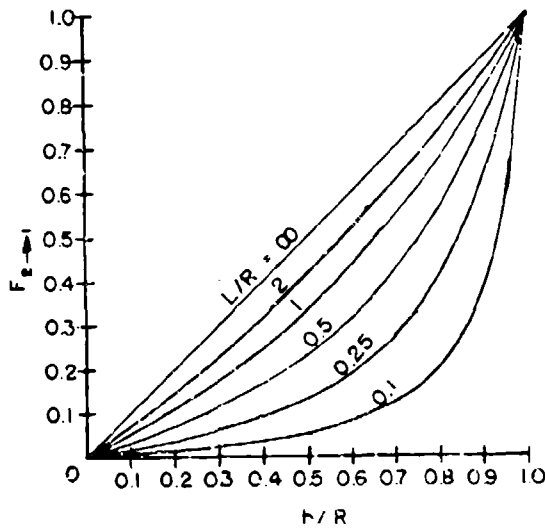
Limiting values:

$$\begin{array}{ll} L \rightarrow 0: & \begin{array}{l} F_{2 \rightarrow 1} = 0 \\ F_{2 \rightarrow 2} = 0 \\ F_{2 \rightarrow 3} = \frac{1}{2} \\ F_{3 \rightarrow 4} = 1 \end{array} \\ L \rightarrow \infty: & \begin{array}{l} F_{2 \rightarrow 1} = r/R \\ F_{2 \rightarrow 2} = 1 - r/R \\ F_{2 \rightarrow 3} = 0 \\ F_{3 \rightarrow 4} = 0 \end{array} \\ r \rightarrow 0: & \begin{array}{l} F_{2 \rightarrow 1} = 0 \\ F_{2 \rightarrow 2} = 1 - \frac{1}{2} \left(\sqrt{4 + \frac{L^2}{R^2}} - \frac{L}{R} \right) \\ F_{2 \rightarrow 3} = \frac{1}{2} \left(\sqrt{4 + \frac{L^2}{R^2}} - \frac{L}{R} \right) \\ F_{3 \rightarrow 4} = 1 - \frac{1}{2} \left(\frac{L}{R} \sqrt{4 + \frac{L^2}{R^2}} - \frac{L^2}{R^2} \right) \end{array} \end{array}$$

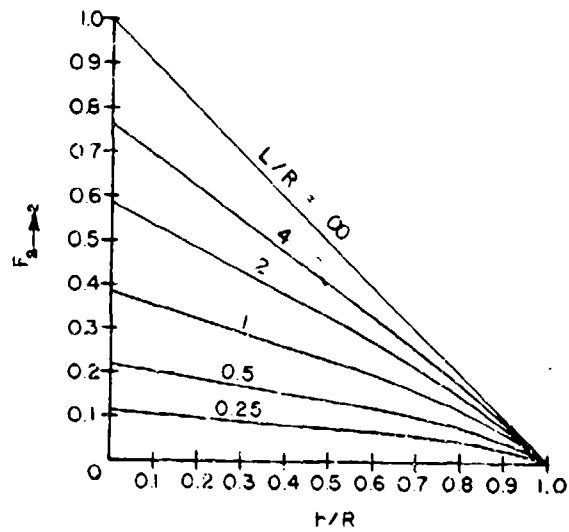
* Evaluated with L replaced by W.

** Values calculated from this expression are in agreement with values obtained by approximate numerical methods by Hamilton and Morgan (loc.cit.).

OUTER CYLINDER TO INNER CYLINDER



OUTER CYLINDER TO ITSELF



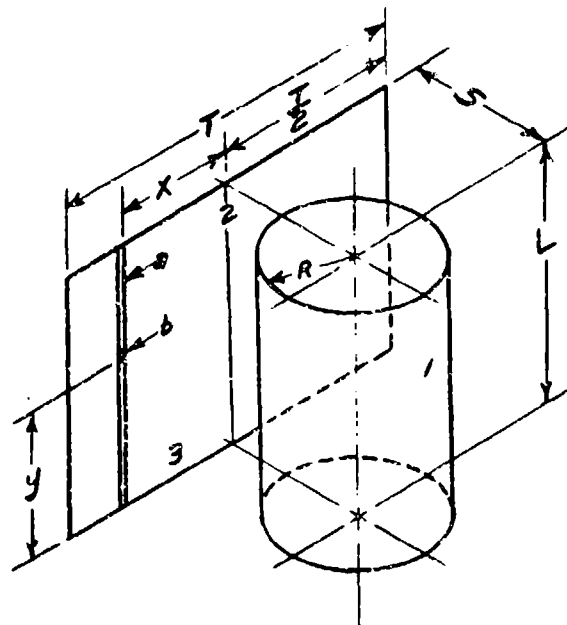
VI Concentric cylinders of equal length, one inside the other

$$\begin{aligned}
 r \longrightarrow R: \quad & F_{2 \rightarrow 1} = 1 \\
 & F_{2 \rightarrow 2} = 0 \\
 & F_{2 \rightarrow 3} = 0 \\
 & F_{3 \rightarrow 4} = 0
 \end{aligned}$$

$$\begin{aligned}
 & W \rightarrow L \\
 \text{or } W \rightarrow 0: \quad F_{d \rightarrow 1} &= \frac{r}{R} \left\{ 1 - \frac{1}{\pi} \left[\cos^{-1} \frac{L^2 - R^2 + r^2}{L^2 + R^2 - r^2} - \frac{L}{r} \left[\frac{L^2 + R^2 + r^2}{\sqrt{(L^2 + R^2 + r^2)^2 - 4r^2R^2}} - \frac{r(L^2 - R^2 + r^2)}{R(L^2 + R^2 - r^2)} - \frac{r}{R} \right] \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 & R \rightarrow L \\
 \text{or } R \rightarrow 0: \quad F_{d \rightarrow 2} &= 1 - \frac{r}{R} + \frac{L}{2R} - \frac{L^2 + 2R^2}{2R\sqrt{L^2 + 4R^2}} + \frac{1}{\pi} \left\{ \frac{2r}{q} \tan^{-1} \frac{2\sqrt{R^2 - r^2}}{L} + \frac{L}{R} \sin^{-1} \left(1 - \frac{2r^2}{R^2} \right) \right. \\
 & \quad \left. - \frac{L^2 + 2R^2}{R\sqrt{L^2 + 4R^2}} \sin^{-1} \frac{4(R^2 - r^2) + \frac{L^2}{R^2}(R^2 - 2r^2)}{L^2 + 4(R^2 - r^2)} \right\}
 \end{aligned}$$

VII Cylinder and Plane of Equal Length Parallel to Cylinder Axis, Plane Outside Cylinder.



- 1 curved surface of cylinder
 2 plane
 3 one-half of plane
 R radius of cylinder
 L height of cylinder and plane
 S distance from cylinder axis to plane
 T length of plane
 a vertical differential strip of 2
 b differential element of a
 x distance from a to center of 2
 y distance from bottom edge of plane to b
 $S \neq R$

$$F_{b \rightarrow 2} = \frac{SR}{S^2 + x^2} \left\{ 2 - \frac{1}{\pi} \left(\cos^{-1} \frac{y^2 - S^2 - x^2 + R^2}{y^2 + S^2 + x^2 - R^2} + \cos^{-1} \frac{(L-y)^2 - S^2 - x^2 + R^2}{(L-y)^2 + S^2 + x^2 - R^2} \right. \right. \\
- \frac{y}{R} \left[\frac{y^2 + S^2 + x^2 + R^2}{\sqrt{(y^2 + S^2 + x^2 - R^2)^2 + 4y^2 R^2}} \cos^{-1} \frac{R(y^2 - S^2 - x^2 + R^2)}{\sqrt{S^2 + x^2}(y^2 + S^2 + x^2 - R^2)} \right] \\
- \left. \left(\frac{L-y}{R} \right) \left[\frac{(L-y)^2 + S^2 + x^2 + R^2}{\sqrt{[(L-y)^2 + S^2 + x^2 - R^2]^2 + 4(L-y)^2 R^2}} \cos^{-1} \frac{R[(L-y)^2 - S^2 - x^2 + R^2]}{\sqrt{S^2 + x^2}[(L-y)^2 + S^2 + x^2 - R^2]} \right] \right. \\
\left. \left. + \frac{L}{R} \cos^{-1} \frac{R}{\sqrt{S^2 + x^2}} \right) \right\}$$

$$F_{a \rightarrow 2} = \frac{SR}{S^2 + x^2} \left\{ 1 - \frac{1}{\pi} \left(\cos^{-1} \frac{L^2 - S^2 - x^2 + R^2}{L^2 + S^2 + x^2 - R^2} \right. \right. \\
- \frac{1}{2RL} \left[\sqrt{(L^2 + S^2 + x^2 - R^2)^2 + 4L^2 R^2} \cos^{-1} \frac{R(L^2 - S^2 - x^2 + R^2)}{\sqrt{S^2 + x^2}(L^2 + S^2 + x^2 - R^2)} \right. \\
\left. \left. + (L^2 - S^2 - x^2 + R^2) \sin^{-1} \frac{R}{\sqrt{S^2 + x^2}} - \frac{\pi(L^2 + S^2 + x^2 - R^2)}{2} \right] \right\}$$

$$F_{2 \rightarrow 1} = F_{3 \rightarrow 1} = \frac{2}{\pi} \int_0^{T/2} F_{a \rightarrow 1} dx$$

Limiting values:

$$\begin{aligned} \text{or } y \rightarrow 0 \text{ or } y \rightarrow L : F_{b \rightarrow 1} &= \frac{SR}{S^2 + x^2} \left\{ 1 - \frac{1}{\pi} \left[\cos^{-1} \frac{L^2 - S^2 - x^2 + R^2}{L^2 + S^2 + x^2 - R^2} \right. \right. \\ &\quad - \frac{L}{R} \left[\frac{L^2 + S^2 + x^2 + R^2}{\sqrt{(L^2 + S^2 + x^2 - R^2)^2 + 4L^2 R^2}} \cos^{-1} \frac{R}{\sqrt{S^2 + x^2}} \frac{(L^2 - S^2 - x^2 + R^2)}{(L^2 + S^2 + x^2 - R^2)} \right. \\ &\quad \left. \left. \left. - \cos^{-1} \frac{R}{\sqrt{S^2 + x^2}} \right] \right] \right\} \end{aligned}$$

$$L \rightarrow \infty : F_{b \rightarrow 1} = \frac{RS}{S^2 + x^2}$$

$$F_{a \rightarrow 1} = \frac{RS}{S^2 + x^2}$$

$$F_{2 \rightarrow 1} = F_{3 \rightarrow 1} = \frac{2R \tan^{-1} T}{T}$$

$$L \rightarrow 0 : F_{b \rightarrow 1} = 0$$

$$F_{a \rightarrow 1} = 0$$

$$F_{2 \rightarrow 1} = 0$$

$$S \rightarrow \infty : F_{b \rightarrow 1} = 0$$

$$F_{a \rightarrow 1} = 0$$

$$F_{2 \rightarrow 1} = 0$$

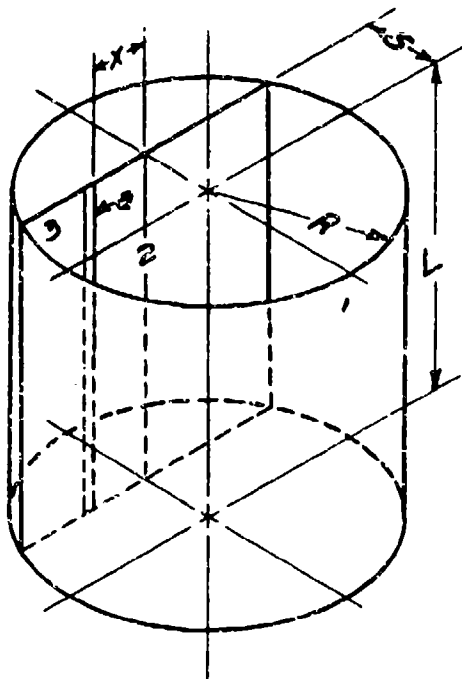
$$\begin{aligned} S \rightarrow R : F_{b \rightarrow 1} &= \left(\frac{R^2}{R^2 + x^2} \right) \left\{ 2 - \frac{1}{\pi} \left[\cos^{-1} \frac{y^2 - x^2}{y^2 + x^2} + \cos^{-1} \frac{(L-y)^2 - x^2}{(L-y)^2 + x^2} \right. \right. \\ &\quad - \frac{y}{R} \left[\frac{y^2 + x^2 + 2R^2}{\sqrt{(y^2 + x^2)^2 - 4y^2 R^2}} \cos^{-1} \frac{R}{\sqrt{R^2 + x^2}} \frac{(y^2 - x^2)}{(y^2 + x^2)} \right] \\ &\quad - \frac{L-y}{R} \left[\frac{(L-y)^2 + x^2 + 2R^2}{\sqrt{[(L-y)^2 + x^2]^2 - 4(L-y)^2 R^2}} \cos^{-1} \frac{R}{\sqrt{R^2 + x^2}} \frac{[(L-y)^2 - x^2]}{[(L-y)^2 + x^2]} \right. \\ &\quad \left. \left. \left. + \frac{L}{R} \cos^{-1} \frac{R}{\sqrt{R^2 + x^2}} \right] \right] \right\} \end{aligned}$$

$$\begin{aligned}
 S \rightarrow R: F_{a \rightarrow 1} &= \frac{R^2}{R^2 + x^2} \left\{ 1 - \frac{1}{\pi} \left[\cos^{-1} \frac{L^2 - x^2}{L^2 + x^2} \right. \right. \\
 &\quad - \frac{1}{2RL} \left\{ \sqrt{(L^2 + x^2)^2 + 4L^2 R^2} \cos^{-1} \frac{R}{\sqrt{R^2 + x^2}} \frac{(L^2 - x^2)}{(L^2 + x^2)} \right. \\
 &\quad \left. \left. + (L^2 - x^2) \sin^{-1} \frac{R}{\sqrt{R^2 + x^2}} - \frac{\pi}{2} (L^2 + x^2) \right\} \right\}
 \end{aligned}$$

$$\begin{aligned}
 R \rightarrow O: F_{b \rightarrow 1} &= 0 \\
 F_{a \rightarrow 1} &= 0 \\
 F_{2 \rightarrow 1} &= 0 \\
 T \rightarrow O: F_{2 \rightarrow 1} &= F_{a \rightarrow 1}^*
 \end{aligned}$$

* evaluated at $x = 0$

VIII Cylinder and Plane of Equal Length Parallel to Cylinder Axis, Plane inside Cylinder



- 1 vertical inner surface of cylinder
- 2 side of plane facing 1
- 3 one half of plane
- R radius of cylinder
- L height of cylinder and plane
- S distance from cylinder axis to plane (taken positive as shown)
- a vertical differential strip of 2
- x distance from a to center of plane

$$\begin{aligned}
F_{a \rightarrow 1} = 1 - \frac{1}{\pi} & \left\{ \tan^{-1} \frac{\sqrt{R^2 - S^2} + x}{L} + \tan^{-1} \frac{\sqrt{R^2 - S^2} - x}{L} + \frac{x(R^2 - S^2 - x^2)}{4L(x^2 + S^2)} \ln \left[\frac{(\sqrt{R^2 - S^2} + x)^2 (\sqrt{R^2 - S^2} - x)^2 + L^2}{(\sqrt{R^2 - S^2} - x)^2 (\sqrt{R^2 - S^2} + x)^2 + L^2} \right] \right. \\
& + \frac{xL}{4(x^2 + S^2)} \ln \frac{(\sqrt{R^2 - S^2} - x)^2 + L^2}{(\sqrt{R^2 - S^2} + x)^2 + L^2} + \frac{S}{4L(x^2 + S^2)} \left\{ L^2 \left[\cos^{-1} \frac{S^2 - x\sqrt{R^2 - S^2}}{R\sqrt{S^2 + x^2}} \right. \right. \\
& \left. \left. + \cos^{-1} \frac{S^2 + x\sqrt{R^2 - S^2}}{R\sqrt{S^2 + x^2}} - 2\pi \right] \right. \\
& + \sqrt{L^2 + R^2 + S^2 + x^2} - 4R^2(x^2 + S^2) \left[\cos^{-1} \frac{2R^2x^2 + S^2(R^2 - S^2 - x^2 - L^2) + x(R^2 + S^2 + x^2 + L^2)\sqrt{R^2 - S^2}}{R\sqrt{S^2 + x^2}[(\sqrt{R^2 - S^2} + x)^2 + L^2]} \right. \\
& \left. + \cos^{-1} \frac{2R^2x^2 + S^2(R^2 - S^2 - x^2 - L^2) - x(R^2 + S^2 + x^2 + L^2)\sqrt{R^2 - S^2}}{R\sqrt{S^2 + x^2}[(\sqrt{R^2 - S^2} - x)^2 + L^2]} \right] \\
& - (R^2 - S^2 - x^2) \left[\cos^{-1} \frac{S^2(R^2 - S^2 - x^2) + 2R^2x^2 - x(R^2 + S^2 + x^2)\sqrt{R^2 - S^2}}{R\sqrt{S^2 + x^2}(\sqrt{R^2 - S^2} - x)^2} \right. \\
& \left. + \cos^{-1} \frac{S^2(R^2 - S^2 - x^2) + 2R^2x^2 + x(R^2 + S^2 + x^2)\sqrt{R^2 - S^2}}{R\sqrt{S^2 + x^2}(\sqrt{R^2 - S^2} + x)^2} \right] \left. \right\}
\end{aligned}$$

$$F_{2 \rightarrow 1} = F_{3 \rightarrow 1} = \frac{1}{\sqrt{R^2 - S^2}} \int_0^{\sqrt{R^2 - S^2}} F_{a \rightarrow 1} dx$$

Limiting values:

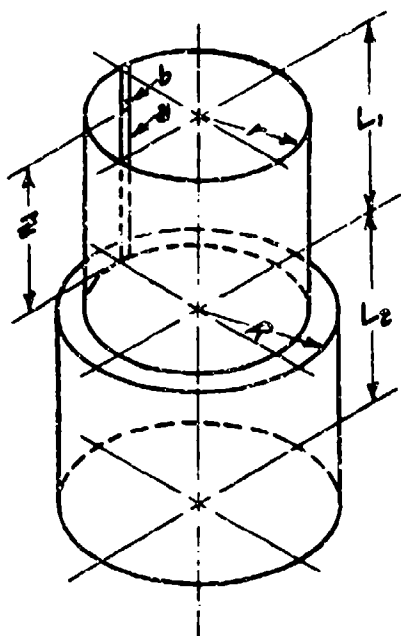
$$\begin{aligned}
x \rightarrow 0: F_{a \rightarrow 1} = 1 - \frac{2}{\pi} & \left\{ \tan^{-1} \frac{\sqrt{R^2 - S^2}}{L} + \frac{1}{4LS} \left[(L^2 - R^2 + S^2) \cos^{-1} \frac{S}{R} - \pi L^2 \right. \right. \\
& \left. \left. + \sqrt{(L^2 + R^2 + S^2)^2 - 4R^2S^2} \cos^{-1} \frac{S(R^2 - S^2 - L^2)}{R(\sqrt{R^2 - S^2} + L)} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
S \rightarrow 0: F_{a \rightarrow 1} = 1 - \frac{1}{\pi} & \left\{ \tan^{-1} \frac{R + x}{L} + \tan^{-1} \frac{R - x}{L} + \frac{R^2 - x^2}{4Lx} \ln \left[\frac{(R + x)^2 ((R - x)^2 + L^2)}{(R - x)^2 ((R + x)^2 + L^2)} \right] \right. \\
& \left. + \frac{L}{4x} \ln \frac{(R - x)^2 + L^2}{(R + x)^2 + L^2} \right\}
\end{aligned}$$

$$\begin{aligned}
& x \rightarrow 0 \\
& \text{and } S \rightarrow R: F_{a \rightarrow 1} = \frac{1}{4} \left(\frac{\sqrt{L^2 + 4R^2}}{R} - \frac{L}{R} \right)
\end{aligned}$$

$$\begin{aligned}
& L \rightarrow \infty \quad F_{a \rightarrow 1} = 1 \\
& \quad \quad \quad F_{2 \rightarrow 1} = 1
\end{aligned}$$

IX Two Concentric Cylinders of Different Radii, One Atop the Other



- 1 inner curved surface of top cylinder
- 2 inner curved surface of bottom cylinder
- 3 top of cylinder 1
- 4 base of cylinder 2
- R radius of 2
- r radius of 1
- L₁ height of 1
- L₂ height of 2
- a differential vertical element of 1
- b differential element of a
- z height of b from base of 1
- R ≥ r

A. $\frac{L_2}{L_1} \leq \frac{1}{2} \left(\frac{R}{r} - 1 \right)$

(2 receives no direct radiation from 3)

$$F_{1 \rightarrow 2} = \frac{r}{4L_1} \left[1 - \frac{R^2 + L_1^2}{r^2} + \sqrt{\left(1 + \frac{R^2 + L_1^2}{r^2} \right)^2 - 4 \frac{R^2}{r^2}} \right]$$

$$F_{1 \rightarrow 4} = \frac{1}{4} \left(\sqrt{4 + \frac{L_1^2}{r^2}} - \frac{L_1}{r} \right) - \frac{r}{4L_1} \left[1 - \frac{R^2 + L_1^2}{r^2} + \sqrt{\left(1 + \frac{R^2 + L_1^2}{r^2} \right)^2 - 4 \frac{R^2}{r^2}} \right]$$

B. $\frac{L_2}{L_1} \geq \frac{1}{2} \left(\frac{R}{r} - 1 \right)$

$$F_{b \rightarrow 4} = \frac{1}{\pi} \left\{ \frac{z^2 + 2r^2}{r\sqrt{z^2 + 4r^2}} \tan^{-1} \frac{\sqrt{z^2 + 4r^2} \sqrt{R^2 - r^2}}{\sqrt{r^2(2L_2 + z)^2 - R^2 z^2}} - \frac{z \sin^{-1} \frac{z \sqrt{R^2 - r^2}}{2r\sqrt{L_2(L_2 + z)}}}{r} \right. \\ + \left(\frac{z + L_2}{2r} \right) \left(\frac{R^2 + r^2 + (z + L_2)^2}{[z^2 + L_2^2 + R^2 + r^2]^2 - 4R^2 r^2} \cos^{-1} \frac{2r^2 L_2 [R^2 - r^2 - (z + L_2)^2] + z(R^2 - r^2)[R^2 - r^2 + (z + L_2)^2]}{2Rr(z + L_2)[L_2(z + L_2) + R^2 - r^2]} \right. \\ \left. \left. - \cos^{-1} \frac{z \sqrt{R^2 - r^2} - 2L_2 r^2}{2RrL_2} \right) \right\}$$

$$F_{1 \rightarrow 4} = F_{a \rightarrow 4} - \frac{\int_{L_1}^{L_2} F_{b \rightarrow 4} dZ}{L_1}$$

$$F_{1 \rightarrow 2} = \frac{1}{4} \left[\sqrt{4 + \frac{L_1^2}{R^2}} - \frac{L_1}{R} \right] - F_{a \rightarrow 4}$$

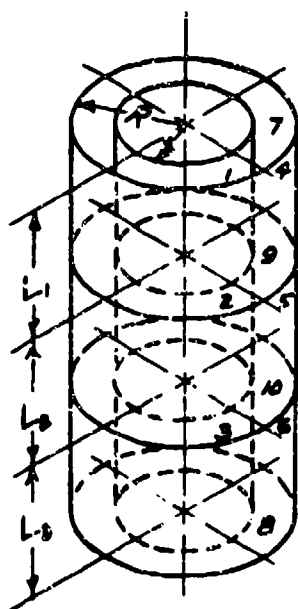
Limiting values:

$$Z \rightarrow 0: F_{b \rightarrow 4} = \frac{1}{\pi} \left\{ \tan^{-1} \frac{\sqrt{R^2 - r^2}}{L_2} + \frac{L_2}{2r} \left[\frac{R^2 + r^2 + L_2^2}{\sqrt{(L_2^2 + R^2 + r^2)^2 - 4R^2 r^2}} \cos^{-1} \frac{R(R^2 - r^2 - L_2^2)}{R(L_2^2 + R^2 - r^2)} - \cos^{-1} \left(-\frac{r}{R} \right) \right] \right\}$$

$$\text{and } r \rightarrow R: F_{b \rightarrow 4} = \frac{L_2^2 + 2R^2}{2R\sqrt{L_2^2 + 4R^2}} - \frac{L_2}{2R}$$

$$r \rightarrow R: F_{1 \rightarrow 4} = \frac{1}{4} \left[\left(\frac{L_1 + L_2}{L_1} \right) \sqrt{4 + \frac{(L_1 + L_2)^2}{R^2}} - \frac{L_1 + 2L_2}{R} - \frac{L_2}{L_1} \sqrt{4 + \frac{L_2^2}{R^2}} \right]$$

II Two Concentric Cylinders of Unequal Length. One Enclosed by the Other



- 1 exterior surface of top inner cylinder
- 2 exterior surface of middle inner cylinder
- 3 exterior surface of bottom inner cylinder
- 4 interior surface of top outer cylinder
- 5 interior surface of middle outer cylinder
- 6 interior surface of bottom outer cylinder
- 7 annulus between tops of 1 and 4
- 8 annulus between bottoms of 3 and 6
- 9 annulus between bottoms of 1 and 4
- 10 annulus between bottoms of 2 and 5
- R radius of outer cylinders
- r radius of inner cylinders
- L₁ height of top cylinders
- L₂ height of middle cylinders
- L₃ height of bottom cylinders
- 9 and 10 are used for calculation purposes only and do not shield radiation between cylinders.

$$F_{2 \rightarrow 4 + 5 + 6} = 1 - \frac{R^2 - r^2}{2rL_2} \left[F_{7 \rightarrow 1 + 2} + F_{8 \rightarrow 2 + 3} - F_{7 \rightarrow 1} - F_{8 \rightarrow 3} \right]$$

$$F_{5 \rightarrow 1 + 2 + 3} = F_{5 \rightarrow 2} + \frac{1}{RL_2} \left[r(L_1 F_{1 \rightarrow 9} + L_3 F_{3 \rightarrow 10}) - \frac{R^2 - r^2}{2} (F_{10 \rightarrow 1 + 2} - F_{9 \rightarrow 2 + 3} - F_{10 \rightarrow 2} - F_{9 \rightarrow 2}) \right]$$

F's on the right side of the above equations may be evaluated from VI (two concentric cylinders of equal length).